ELSEVIER
Theoretical Computer Science 255 (2001) 679–686
Theoretical Computer Science
www.elsevier.com/locate/tcs

Note

Some linear recurrences and their combinatorial interpretation by means of regular languages ☆

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Received May 2000; accepted September 2000
Communicated by M. Nivat

Abstract

In this paper we apply ECO method and the concept of numeration systems to give a combinatorial interpretation to linear recurrences of the kind $a_n = ka_{n-1} + ha_{n-2}$, where $k > |h| \geqslant 0$. In particular, we define a language $\mathscr{L}$ such that the words of $\mathscr{L}$ having length n satisfy the recurrence, and then we describe a recursive construction for this language, according to the ECO method, and the corresponding finite succession rule. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Linear recurrence relations; Numeration systems; Succession rules; Regular languages

1. Introduction

We provide a combinatorial interpretation to a general class of two terms linear recurrences defining nonnegative integer sequences, which include some of the most frequently occurring ones; the recurrences we deal with have the form

$$a_{-1} = 0,$$
$$a_0 = 1,$$
$$a_n = ka_{n-1} + ha_{n-2}, \quad n > 1,$$

where $k \in \mathbb{N}^+$ and h is an integer such that $k > |h|$. In this way, we wish to provide a general solution to a problem arising from [3], where Bonin et al. asked for the

☆ This work was partially supported by MURST project: *Modelli di calcolo innovativi: metodi sintattici e combinatori.*

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PII: S0304-3975(00)00398-4

combinatorial interpretation of the recurrence $f_{n+1}=6f_n - f_{n-1}$, with $f_0=1$, $f_1=7$ (sequence M4423 of [7]); Barcucci et al. [1] gave the first answer using a regular language defined so that the number of words having length n is equal to f_{n+1}. Sulanke [8] and Pergola and Pinzani [6] gave a combinatorial interpretation of the recurrence in terms of total area of Schröder paths.

We begin by briefly recalling some concepts related to the ECO method. It constructs a class of combinatorial objects $\mathcal{O}$ by means of an operator ϑ which performs a "local expansion" on the objects. We refer to [2] for further details, proofs and definitions. Let p be a *discriminating parameter* on $\mathcal{O}$, that is $p: \mathcal{O} \to \mathbb{N}^+$, such that $|\mathcal{O}_n| = |\{O \in \mathcal{O}: p(O)=n\}|$ is finite. An operator ϑ on the class $\mathcal{O}$ is a function from $\mathcal{O}_n$ to $2^{\mathcal{O}_{n+1}}$, where $2^{\mathcal{O}_{n+1}}$ is the power set of $\mathcal{O}_{n+1}$.

Proposition 1.1. *Let ϑ be an operator on $\mathcal{O}$. If ϑ satisfies the following conditions*:
1. *for each $O' \in \mathcal{O}_{n+1}$, there exists $O \in \mathcal{O}_n$ such that $O' \in \vartheta(O)$,*
2. *for each $O, O' \in \mathcal{O}_n$ with $O \neq O'$, $\vartheta(O) \cap \vartheta(O') = \emptyset$,*

then the family of sets $\mathcal{F}_{n+1} = \{\vartheta(O): O \in \mathcal{O}_n\}$ is a partition of $\mathcal{O}_{n+1}$.

Once the parameter p is fixed, if we are able to define an operator ϑ which satisfies conditions 1 and 2, then Proposition 1.1 allows us to construct each object $O' \in \mathcal{O}_{n+1}$ from an object $O \in \mathcal{O}_n$, and each object $O' \in \mathcal{O}_{n+1}$ is obtained from one and only one $O \in \mathcal{O}_n$.

The recursive construction determined by ϑ can be described by introducing the concept of *generating tree* [4]: it is a rooted tree whose vertices are objects of $\mathcal{O}$. The objects having the same value of the parameter p lie at the same level, and the sons of an object are the objects it produces through ϑ. A generating tree can be described by a succession rule Ω:

$$
\begin{aligned}
&(b)\\
&(h) \rightsquigarrow (c_1)(c_2)\cdots(c_h),
\end{aligned}
$$

meaning that the root object has b sons, and the h objects $O'_1, \ldots, O'_h$, produced by an object O are such that $|\vartheta(O'_i)| = c_i$, $1 \leqslant i \leqslant h$.

2. Preliminaries

Let $\{a_n\}_{n\geqslant 0}$ be a sequence of positive integers such that $a_0=1$ and $a_n < a_{n+1}$ for $n \in \mathbb{N}$; let N be any non-negative integer and a_n the largest number in the sequence not exceeding N (except $n=0$ if $N=0$). If we divide N by a_n and iterate, we have

$$
\begin{array}{ll}
N = d_n a_n + r_n & 0 \leqslant r_n < a_n,\\
r_n = d_{n-1} a_{n-1} + r_{n-1} & 0 \leqslant r_{n-1} < a_{n-1},\\
\vdots \quad \vdots \quad \vdots & \vdots \quad \vdots
\end{array}
$$

$$r_2 = d_1a_1 + r_1 \quad 0 \leqslant r_1 < a_1,$$
$$r_1 = d_0a_0.$$

Therefore, $N = d_na_n + d_{n-1}a_{n-1} + \cdots + d_0a_0$. It is the *representation* of N in the *numeration system* $\{a_n\}_{n\geqslant 0}$. A complete proof of the following theorem is given in [5].

Theorem 2.1. *Given $\{a_n\}_{n\geqslant 0}$, let $1 = a_0 < a_1 < \cdots < a_i < \cdots$ be an integer sequence. Any non-negative integer N has precisely one representation in the system $\{a_n\}_{n\geqslant 0}$, of the form $N = \sum_{i=0}^{n} d_ia_i$, where d_i are non-negative integers satisfying*

$$a_id_i + a_{i-1}d_{i-1} + \cdots + a_0d_0 < a_{i+1} \quad (i \geqslant 0). \tag{1}$$

In this paper, we aim at considering the sequence $\{a_n\}_{n\geqslant 0}$ described by the recurrence relation

$$a_{-1} = 0,$$
$$a_0 = 1,$$
$$a_n = ka_{n-1} + ha_{n-2}, \quad n > 1,$$

where $k \in \mathbb{N}^+$ and $h \in \mathbb{Z}$. It should be clear that the condition required for the terms of the sequence $\{a_n\}_{n\geqslant 0}$ to be non-negative is $k^2 + 4h \geqslant 0$, and this also implies that the sequence is strictly increasing, except for the trivial case $a_n = a_{n-1}$.

3. The interpretation of the recurrence

Let us now define the sets $\mathscr{L}_n$:

$$\mathscr{L}_0 = \{\varepsilon\},$$
$$\mathscr{L}_n = \{d_{n-1}\dots d_0 : d_{n-1}\dots d_0 \text{ is the representation of } m < a_n \text{ in the system } \{a_n\}_{n\geqslant 0}\}.$$

Moreover, let $\mathscr{L} = \bigcup_n \mathscr{L}_n$. The following properties trivially hold:
1. the sets $\mathscr{L}_n$ are mutually disjoint, and each $\mathscr{L}_n$ contains all the n-length words in $\mathscr{L}$;
2. for all n, $|\mathscr{L}_n| = a_n$.

Example 3.1. Let $\{a_n\}_{n\geqslant 0}$ denote the sequence of *Pell numbers* (sequence M1413 of [7]) defined by the recurrence

$$a_{-1} = 0,$$
$$a_0 = 1,$$
$$a_n = 2a_{n-1} + a_{n-2}, \quad n > 1.$$

Each non-negative integer has a unique representation in the numeration system $\{a_n\}_{n\geqslant 0}$.

29	12	5	2	1	n	29	12	5	2	1	n
				0	0		1	0	2	1	15
				1	1		1	0	2	0	16
			1	0	2		1	1	0	0	17
			1	1	3		1	1	0	1	18
			2	0	4		1	1	1	0	19
		1	0	0	5		1	1	1	1	20
		1	0	1	6		1	1	2	0	21
		1	1	0	7		1	2	0	0	22
		1	1	1	8		1	2	0	1	23
		1	2	0	9		2	0	0	0	24
		2	0	0	10		2	0	0	1	25
		2	0	1	11		2	0	1	0	26
	1	0	0	0	12		2	0	1	1	27
	1	0	0	1	13		2	0	2	0	28
	1	0	1	0	14	1	0	0	0	0	29

We then construct the sets $\mathscr{L}_n$:

$$\mathscr{L}_0 = \{\varepsilon\},$$

$$\mathscr{L}_1 = \{0,1\},$$

$$\mathscr{L}_2 = \{00,01,10,11,20\},$$

$$\mathscr{L}_3 = \{000,001,010,011,020,100,101,110,111,120,200,201\},$$

$$\mathscr{L}_3 = \{0000,0001,0010,0011,0020,0100,0101,0110,0111,0120,0200,0201,1000,$$
$$1001,1010,1021,1020,1100,1101,1110,1111,1120,1200,1201,2000,2001,$$
$$2010,2011,2020\}.$$

We observe that $\mathscr{L}$ is the set of all words on $\{0,1,2\}$ such that each 2 is followed by 0, and represented by the regular expression $(0 \vee 1 \vee 20)^{\star}$.

Let us denote by Σ the set of terminal symbols for $\mathscr{L}$; for each $n\in\mathbb{N}$ the alphabet for $\mathscr{L}_n$ is simply $\{0,1,\ldots,\lfloor (a_{n+1}-1)/a_n\rfloor\}$. Moreover, since the recurrence we deal with is two-termed and satisfies $k^2+4h\geqslant 0$, we state that $\Sigma=\{0,1,\ldots,\alpha\}$, with

$$\alpha = \max\left\{a_1-1, \left\lfloor \frac{a_2-1}{a_1}\right\rfloor\right\} = \max\left\{k-1, \left\lfloor \frac{k^2+h-1}{k}\right\rfloor\right\}.$$

The set $\mathscr{L}$ is a language over the alphabet Σ, that is, a combinatorial interpretation for the recurrence relation considered; the general problem now is to define $\mathscr{L}$ by

means of a (possibly regular) grammar, and to give a recursive construction for $\mathscr{L}$ according to the ECO method. We solve this problem in two particular cases, which include the most frequently occurring two-terms recurrences
1. $k \geqslant h \geqslant 0$;
2. $k > -h \geqslant 0$.

Before proceeding, we underline that both these cases imply the required condition $k^2 + 4h \geqslant 0$ for the terms a_n to be all positive, and $\lfloor (k^2 + h - 1)/k \rfloor \geqslant k - 1$, thus we ensure that $\Sigma = \{0, 1, \ldots, \lfloor (k^2 + h - 1)/k \rfloor\}$.

1. The set of terminal symbols is then $\Sigma = \{0, 1, \ldots, k\}$; the language $\mathscr{L}$ can be described as the set of all the words $w \in \Sigma^\star$, $w = d_r \ldots d_0$, with $d_i \in \Sigma$, and such that if $d_i = k$, then $d_{i-1} < h$, for each $i = 1, \ldots, r$, and $d_0 \neq k$. The following unambiguous regular grammar generates $\mathscr{L}$:

$$S \to T0|T1|\cdots|T(h-1)|Sh|\cdots|S(k-1)|\varepsilon,$$

$$T \to T0|T1|\cdots|T(h-1)|Sh|\cdots|S(k)|\varepsilon.$$

It is now possible to define an operator ϑ satisfying conditions 1 and 2 of Proposition 1.1 which constructs all the words of $\mathscr{L}_n$ from the words of $\mathscr{L}_{n-1}$. Let w be a word of $\mathscr{L}_{n-1}$, the operator ϑ works as follows:
- if the first symbol of w is $j \leqslant h - 1$, then ϑ inserts a symbol $s = 0, \ldots, k$ on the left of w;
- if w is ε or begins by some $j \geqslant h$, then ϑ inserts on the left of w a symbol $s = 0, \ldots, k - 1$.

The recursive construction defined by ϑ can be represented by means of the succession rule

$$\begin{aligned}&(k),\\ &(k) \rightsquigarrow (k)^{k-h}(k+1)^h,\\ &(k+1) \rightsquigarrow (k)^{k-h+1}(k+1)^h.\end{aligned}$$

Example 3.2. Consider the recurrence relation introduced in Example 3.1 that describes the sequence of Pell numbers; here the language $\mathscr{L}$ is generated by the grammar

$$S \to T0|S1|\varepsilon,$$

$$T \to T0|S1|S2|\varepsilon.$$

The operator ϑ defines a recursive construction for $\mathscr{L}$, and its generating tree, whose first levels are shown in Fig. 1, can be represented by means of the following succession rule

$$\begin{aligned}&(2),\\ &(2) \rightsquigarrow (2)(3),\\ &(3) \rightsquigarrow (2)(2)(3).\end{aligned}$$

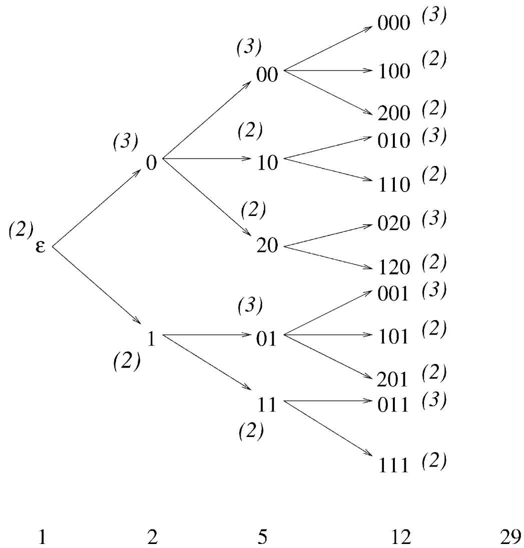

Fig. 1. The first levels of ϑ generating tree.

2. In this case the set of terminal symbols is $\Sigma = \{0, 1, \ldots, k-1\}$; besides, the recurrence relation can be rewritten in a more useful way, involving only non-negative summands

$$\begin{aligned}
a_n &= ka_{n-1} - ha_{n-2} \\
&= (k-1)a_{n-1} + a_{n-1} - ha_{n-2} \\
&= (k-1)a_{n-1} + (k-h-1)a_{n-2} + a_{n-2} - ha_{n-3} \\
&\cdots \quad \cdots \quad \cdots \quad \cdots \\
&= (k-1)a_{n-1} + (k-h-1)a_{n-2} + \cdots + (k-h-1)a_1 + (k-h),
\end{aligned}$$

where $k-h < a_1 = k$, and $n > 1$. As a particular case we have the recurrences of the form $a_n = ka_{n-1} - (k-1)a_{n-2}$, that can therefore be written as $a_n = (k-1)a_{n-1} + 1$. Now it is easy to recognize the language $\mathscr{L}$, as the set of words $w = u_r \ldots u_0 \in \Sigma^\star$ such that if $u_i = k-1$ then $u_{i-1} \leqslant k-h-1$, and if $u_{i-1} = u_{i-2} = \cdots = u_j = k-h-1$, $j > 0$, then $u_{j-1} \leqslant k-h-1$. A generating unambiguous regular grammar for $\mathscr{L}$ is

$$S \to S0|S1|\cdots|S(k-h-1)|T(k-h)|\cdots|T(k-1)|\varepsilon,$$

$$T \to S0|S1|\cdots|S(k-h-2)|T(k-h-1)|\cdots|T(k-2)|\varepsilon.$$

The operator ϑ describing the recursive growth of $\mathscr{L}$ works on $w \in \mathscr{L}$ as follows:

- if w is the empty word or the first symbol of w is $j \leqslant k-h-1$, then ϑ inserts a symbol $s = 0, \ldots, k-1$ on the left of w;

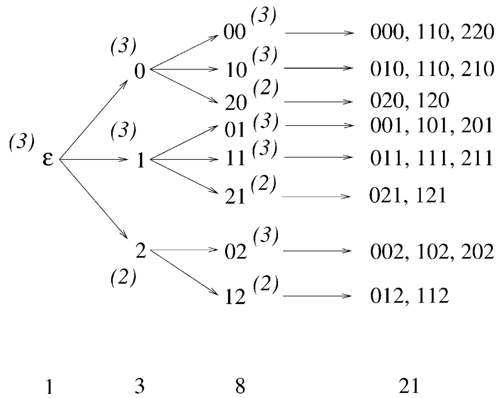

Fig. 2. The first levels of ϑ generating tree for $\mathscr{L}$.

- if the first symbol of w is $j > k-h-1$, then ϑ inserts a symbol $s = 0, \dots, k-2$ on the left of w.

  The succession rule which represents ϑ generating tree is

$$\begin{aligned}&(k),\\&(k-1) \rightsquigarrow (k-1)^h (k)^{k-h-1},\\&(k) \rightsquigarrow (k-1)^h (k)^{k-h}.\end{aligned}$$

Example 3.3. Let $\{a_n\}_n$ denote the sequence of *odd index Fibonacci numbers*, (sequence M2741 of [7]) described by the recurrence relation

$$\begin{aligned}&a_{-1} = 0,\\&a_0 = 1,\\&a_n = 3a_{n-1} - a_{n-2}, \quad n > 1.\end{aligned}$$

According to the previously defined method, we can rewrite the recurrence as $a_n = 2a_{n-1} + a_{n-2} + \cdots + a_1 + 2$, for $n > 1$; now $\mathscr{L}$ is the language defined by the grammar

$$\begin{aligned}&S \to S0|S1|T2|\varepsilon,\\&T \to S0|T1|\varepsilon.\end{aligned}$$

The succession rule which represents the recursive construction of $\mathscr{L}$ obtained by means of the ECO operator ϑ is (see Fig. 2)

$$\begin{aligned}&(3),\\&(2) \rightsquigarrow (2)(3),\\&(3) \rightsquigarrow (2)(3)(3).\end{aligned}$$

Finally we point out that this method can be slightly generalized to all the recurrences of cases 1 and 2, but having arbitrarily chosen initial conditions. For instance, we study the sequence of NSW numbers, defined by the recurrence relation

$$a_0 = 1,$$
$$a_1 = 7,$$
$$a_n = 6a_{n-1} - a_{n-2}, \quad n > 1,$$

it can be written as $a_n = 5a_{n-1} + 4a_{n-2} + \cdots + 4a_1 + 6$, for $n > 1$, and thus the language $\mathscr{L}$ is generated by the unambiguous grammar

$$S \to T0|T1|T2|T3|T4|T5|V6|\varepsilon,$$
$$T \to T0|T1|T2|T3|T4|V5|\varepsilon,$$
$$V \to T0|T1|T2|T3|V4|\varepsilon.$$

The succession rule which describes the generating tree of an ECO operator recursively constructing $\mathscr{L}$ is

$$(7),$$
$$(5) \rightsquigarrow (5)(6)(6)(6)(6),$$
$$(6) \rightsquigarrow (5)(6)(6)(6)(6)(6),$$
$$(7) \rightsquigarrow (5)(6)(6)(6)(6)(6)(6).$$

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